

Limit Theorems for Iteration Stable Tessellations

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Abstract

The intent of this paper is to describe large scale asymptotic geometry of STIT tessellations in \mathbb{R}^d , which form a rather new, rich and flexible class of random tessellations considered in stochastic geometry. For this purpose, martingale tools are combined with second-order formulas proved earlier to establish limit theorems for STIT tessellations. More precisely, a Gaussian functional central limit theorem for the surface increment processes induced by STIT tessellations relative to an initial time moment is shown. As second main result, a central limit theorem for the total edge length/facet surface is obtained, with a normal limit distribution in the planar case and, most interestingly, with a non-normal limit showing up in all higher space dimensions – including the practically relevant spatial case.

Key words: Central Limit Theorem; Iteration/Nesting; Martingale Theory; Random Tessellation; Stochastic Stability; Stochastic Geometry

MSC (2000): Primary: 60D05; Secondary: 60F05; 60G55

1 Introduction and results

Random tessellations (or mosaics) of \mathbb{R}^d are locally finite families of compact convex random polytopes, which have no common interior points, cover the whole space and form a central object studied in stochastic geometry, spatial statistics and related fields, see [1, 2, 4, 7, 8, 9, 10] to name just a few. However, there are only very few mathematically tractable models and the most prominent examples include hyperplane and Voronoi tessellations, where most often the Poisson case is considered. A new class, the so-called STIT tessellations, was introduced recently in [13, 14, 15, 16] and has quickly attracted considerable interest. These tessellations clearly show the potential to become a new fundamental reference model for both, theoretical and practical purposes. Whereas most research on random tessellations in the last decades was about mean values and mean value relations (see [17] for the recent state of the art), modern stochastic geometry focusses on distributional aspects [1, 2, 8, 9, 10] and limit theorems [7]. In contrast to the tessellations studied so far, the STIT model has the additional feature of arising as a result of a spatio-temporal

dynamic construction. From this point of view, limit theorems for STIT tessellations are particularly interesting. As recently pointed out by the first author in [18], we expect that the large scale asymptotic of dynamic models will become of great importance in stochastic geometry.

Let us recall the basic construction of tessellations that arise as a result of repeated cell division. To this end, let Λ be a translation-invariant measure on the space \mathcal{H} of hyperplanes in \mathbb{R}^d . Thus, Λ factorizes under the usual polar identification, i.e.

$$\Lambda := \ell_+ \otimes \mathcal{R}, \quad (1)$$

where ℓ_+ is the Lebesgue measure on the positive real half-axis and where \mathcal{R} is a probability measure on the unit sphere \mathcal{S}_{d-1} . Throughout this paper we always require that the support of \mathcal{R} spans the whole space, i.e. that $\text{span}(\text{supp}(\mathcal{R})) = \mathbb{R}^d$, and we say in this case that Λ is non-degenerate. Further, let $t > 0$ be fixed and let $W \subset \mathbb{R}^d$ be a compact convex window with interior points in which our construction of a random tessellation $Y(t\Lambda, W)$ is carried out. In a first step, we assign to the window W a random lifetime. Upon expiry of its lifetime, the primordial cell W dies and splits into two polyhedral sub-cells W^+ and W^- separated by a hyperplane hitting W , which is chosen according to the normalized distribution Λ . The resulting new cells W^+ and W^- are again assigned independent random lifetimes and the entire construction continues recursively until the deterministic time threshold t is reached (see Figure 1 for an illustration). The described process of recursive cell divisions is called the *MNW-construction* in the sequel and the resulting random tessellation constructed inside W is denoted by $Y(t\Lambda, W)$, as mentioned above.

In order to ensure the Markov property of the above construction in the continuous-time parameter t , we assume from now on that the lifetimes arising in the MNW-construction (including the initial window W) are exponentially distributed. Moreover, we assume that the parameter of the exponentially distributed lifetimes of individual cells $[c]$ equals $\Lambda([c])$, where $[c] := \{H \in \mathcal{H}, H \cap c \neq \emptyset\}$ stands for the collection of hyperplanes hitting cell c . In this special situation, the random tessellations $Y(t\Lambda, W)$ fulfil a stochastic stability property under the operation of iteration of tessellations and are for this reason called random *STIT tessellations*, see Section 2 below for details.

Having studied the first- and second-order properties of STIT tessellations in [20, 21], we consider in this paper the central limit problem. This problem will be considered in two closely related settings, interestingly leading to results of very different qualitative nature. First, we shall focus our interest on the residual length/surface increment arising, respectively, as cumulative length or surface area of the cell-separating $(d-1)$ -polyhedral facets born after a certain fixed time in the course of the MNW-construction recalled in Section 2. In this set-up we shall establish a central limit theorem with a Gaussian limiting variable. Next, we shall pass to the *total* length/surface, taking into account also the polyhedral segments/facets born at the very initial *big bang* stages of the MNW-construction, as descriptively termed in [13]. It turns out that, whereas in dimension 2 the Gaussian convergence, is preserved, this is no more the case for dimensions 3 and higher, where non-Gaussian limits arise. This apparently surprising phenomenon is in fact due to

the influence of the big bang phase in the MNW-construction itself, which is negligible in two dimensions but turns out to be crucial in higher dimensions. As already emphasized above, it is interesting to see that in terms of the time incremental MNW-construction of STIT tessellations our central limit theorems are more reliant on independences arising in construction time than those of spatial nature.

We are now going to describe some of our limit theorems in detail. For W as above, we put $W_R := RW$ for $R > 0$ and let Λ be some fixed translation-invariant measure on \mathcal{H} . Therefore and in order to simplify the notation we will write from now on $Y(t, W)$ instead of $Y(t\Lambda, W)$. Our first limit theorem deals with the total surface area $\text{Vol}_{d-1}(Y(t\Lambda, W))$ of the polyhedral facets constructed by the MNW-construction within the time period $[s_0, 1]$, where $s_0 > 0$ is some positive initial time moment.

Theorem 1 *For each $s_0 > 0$, the random variable*

$$\frac{1}{R^{d/2}}[(\text{Vol}_{d-1}(Y(1, W_R)) - \mathbb{E}\text{Vol}_{d-1}(Y(1, W_R))) \\ - (\text{Vol}_{d-1}(Y(s_0, W_R)) - \mathbb{E}\text{Vol}_{d-1}(Y(s_0, W_R)))]$$

converges, as $R \rightarrow \infty$, in law to $\mathcal{N}(0, V_W(\text{Vol}_{d-1}, \Lambda) \int_{s_0}^1 s^{1-d} ds)$, a normal distribution with mean 0 and variance $V_W(\text{Vol}_{d-1}, \Lambda) \int_{s_0}^1 s^{1-d} ds$, where $V_W(\text{Vol}_{d-1}, \Lambda)$ is explicitly given by (9) or alternatively (17) below.

This statement cannot be extended to $s_0 \downarrow 0$, as would be of interest as potentially yielding a Gaussian limit for the (centred and suitably normalized) total edge length/surface area $\text{Vol}_{d-1}(Y(1, W_R))$. The problem is that the variance integral $V_W(\text{Vol}_{d-1}, \Lambda) \int_{s_0}^t s^{1-d} ds$ diverges at 0. However, this difficulty can be overcome for $d = 2$ but not for $d > 2$. Indeed, in the planar case the asymptotic behaviour of the total edge length turns out to be Gaussian.

Theorem 2 *We have for the STIT tessellation $Y(1)$ in the plane with \mathcal{R} given by the uniform distribution on \mathcal{S}_1 ,*

$$\frac{1}{R\sqrt{\log R}}[\text{Vol}_1(Y(1, W_R)) - \mathbb{E}\text{Vol}_1(Y(1, W_R))] \implies \mathcal{N}(0, \pi \text{Vol}_2(W)),$$

where \implies means convergence in law.

In fact, Theorem 1 and Theorem 2 are direct consequences of our much stronger functional central limit theorems, Theorems 4 and 5 below. Moreover, we will see there that the Gaussian convergence in Theorem 2 is true even for any non-degenerate directional distribution \mathcal{R} .

For space dimensions $d > 2$ we claim that the Gaussian convergence cannot be preserved. Even though we are able to show this fact for all W and translation invariant Λ by establishing non-Gaussian tail decay, for simplicity and in order to keep the argument transparent we only give a proof for an more easily tractable particular case, postponing the study

of more involved properties of the resulting random field to a future paper. We take $W = [0, 1]^d$ and consider

$$\Lambda := \sum_{i=1}^d \int_{-\infty}^{+\infty} \delta_{re_i + e_i^\perp} dr, \quad (2)$$

where e_i , $i = 1, \dots, d$ are vectors of the standard orthonormal basis for \mathbb{R}^d and $\delta_{re_i + e_i^\perp}$ is the unit mass concentrated on the hyperplane orthogonal to e_i in distance r from the origin.

Theorem 3 *For $d > 2$, Λ as in (2) and $W = [0, 1]^d$,*

$$R^{2(d-1)}[\text{Vol}_{d-1}(1, W_R) - \mathbb{E}\text{Vol}_{d-1}(1, W_R)]$$

converges, as $R \rightarrow \infty$, to a non-Gaussian square-integrable random variable $\Xi(W)$ with explicitly known variance given by (23) below.

The plan of the paper is as follows: In the next section we recall some properties of STIT tessellations needed for the proofs of our limit theorems. We also recall there some of the facts from [20, 21] in order to keep the paper self-contained and present the exact statements of our functional central limit theorems. The proofs of our results are the content of Section 3.

We would like to remark that an extended version is available online [19] and, moreover, that the results in the present paper form the basis of our subsequent work [22, 23].

2 Background material and statement of the functional limit theorems

We start by rephrasing some of the properties of the STIT tessellations $Y(t\Lambda, W)$ as defined in the introduction, the proofs of which may be found in [16].

- $Y(t\Lambda, W)$ is consistent in that $Y(t\Lambda, W) \cap V \stackrel{D}{=} Y(t\Lambda, V)$ for convex $V \subset W$ and thus $Y(t\Lambda, W)$ can be extended to a random tessellation $Y(t\Lambda)$ in the whole space \mathbb{R}^d .
- $Y(t\Lambda)$ is a stationary random tessellation, i.e. stochastically translation invariant. If, moreover, Λ is the unit-density isometry-invariant hyperplane measure Λ_{iso} , or equivalently if \mathcal{R} in (1) is the uniform distribution ν_{d-1} on \mathcal{S}_{d-1} , then $Y(t\Lambda_{\text{iso}})$ is even isotropic, i.e. stochastically invariant under rotations around the origin.
- $Y(t\Lambda)$ is **stable** under the operation of **iteration**, denoted by \boxplus . This is to say

$$Y(t\Lambda) \stackrel{D}{=} m(Y((t/m)\Lambda) \boxplus \dots \boxplus Y((t/m)\Lambda)), \quad m = 2, 3, \dots$$

For this reason, $Y(t\Lambda)$ is called a random STIT tessellation in this case. This property was discussed in detail in [20] and we refer to this paper and the references cited therein for further discussion, because our arguments do not explicitly use the stochastic stability but its consequences.

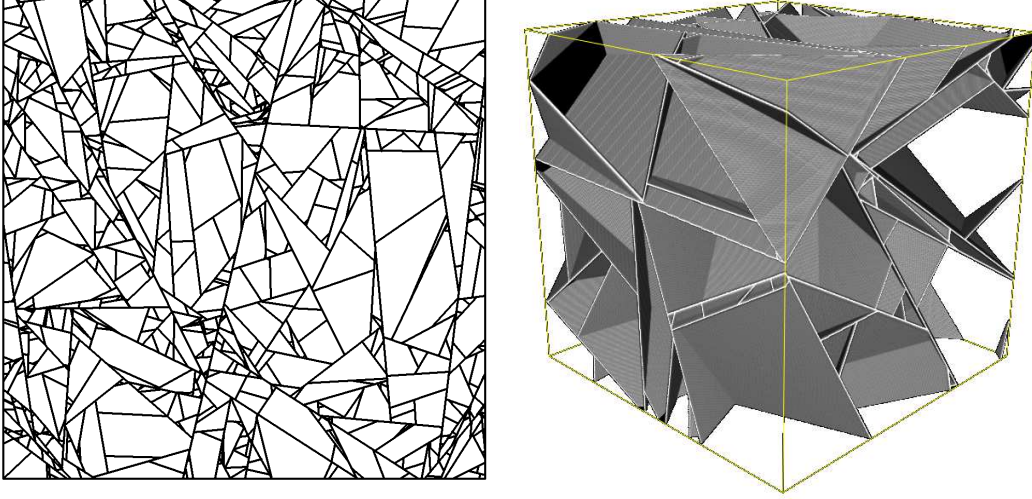


Figure 1: Realizations of a planar and a spatial stationary and isotropic STIT tessellation (kindly provided by Joachim Ohser and Claudia Redenbach)

- The surface density of $Y(t\Lambda)$, i.e. the mean surface area of cell boundaries of $Y(t\Lambda)$ per unit volume equals t . In particular, the mean surface area of facets arising in the MNW-construction during time $[0, t]$ within a compact convex $W \subset \mathbb{R}^d$ with interior points is given by $t \text{Vol}_d(W)$.
- STIT tessellations have the following scaling property:

$$tY(t\Lambda) \stackrel{D}{=} Y(\Lambda),$$

i.e. the tessellation $Y(t\Lambda)$ of surface intensity t upon rescaling by factor t has the same distribution as $Y(\Lambda)$, the STIT tessellation with surface intensity 1.

- STIT tessellations have Poisson typical cells, which is to say, the interior of the typical cell $\text{TypicalCell}(Y(t\Lambda))$ of $Y(t\Lambda)$ coincides in distribution with that of a Poisson hyperplane tessellation $\text{PHT}(t\Lambda)$ with intensity measure $t\Lambda$, see the discussion after Theorem 1 in [20] or [15].

For the nonspecialized reader let us remark that the typical cell of a tessellation is what we get when we choose equiprobably a cell of the tessellation at random out of a ‘large’ observation window. The exact definition makes use of Palm calculus for which we refer to [17]. Moreover, a Poisson hyperplane tessellation $\text{PHT}(t\Lambda)$ is the random subdivision of \mathbb{R}^d induced by a Poisson point process on the space of hyperplanes \mathcal{H} having intensity measure $t\Lambda$.

The finite volume continuous-time incremental MNW-construction of random STIT tessellations, as discussed in the introduction, clearly enjoys the Markov property in the

continuous-time parameter, whence natural martingales arise, which will be of crucial importance for our further considerations. In fact, this observation was the starting point of [20], where a class of martingales associated to STIT tessellations was constructed. In order to streamline our discussion we do not repeat the full theory here, but rephrase the martingale property of two stochastic processes, on which the proofs of our limit theorems are based on. To this end, let Y be some instant of $Y(t\Lambda, W)$ and let $\phi(\cdot)$ be a measurable facet functional of the form

$$\phi(f) := \text{Vol}_{d-1}(f)\zeta(\vec{\mathbf{n}}(f)) \quad (3)$$

with $\vec{\mathbf{n}}(f)$ standing for the unit normal to f and ζ for a bounded measurable function on \mathcal{S}_{d-1} . Moreover, denote the collection of cell-separating $(d-1)$ -dimensional facets, usually referred to as $(d-1)$ -dimensional maximal polytopes, arising in subsequent splits in the MNW-construction by $\text{MaxPolytopes}_{d-1}(Y)$, define $\Sigma_\phi(Y)$ by

$$\Sigma_\phi(Y) := \sum_{f \in \text{MaxPolytopes}_{d-1}(Y)} \phi(f)$$

and $A_\phi(Y)$ to be

$$A_\phi(Y) := \int_{[W]} \sum_{f \in \text{Cells}(Y \cap H)} \phi(f) \Lambda(dH)$$

with $\text{Cells}(Y \cap H)$ standing for the set of $(d-1)$ -dimensional cells of the tessellation $Y \cap H$ induced by the intersection of Y with hyperplane H . Let us further introduce the bar notation $\bar{\Sigma}_\phi(Y)$ for the centred version $\Sigma_\phi(Y) - \mathbb{E}\Sigma_\phi(Y)$ of $\Sigma_\phi(Y)$. Then, we have (see [20, 21])

Proposition 1 *The two stochastic process*

$$\bar{\Sigma}_\phi(Y(t\Lambda, W)) \quad \text{and} \quad \bar{\Sigma}_\phi^2(Y(t\Lambda, W)) - \int_0^t A_{\phi^2}(Y(s\Lambda, W))ds \quad (4)$$

are both \mathfrak{S}_t -martingales, where \mathfrak{S}_t stands for the filtration generated by $(Y(s\Lambda, W))_{0 \leq s \leq t}$.

In particular, see [11, Thm. 4.2], the martingale $\bar{\Sigma}_\phi(Y(t\Lambda, W))$ has its predictable quadratic variation process $\langle \bar{\Sigma}_\phi(Y(\cdot, W)) \rangle$ absolutely continuous and given by

$$\langle \bar{\Sigma}_\phi(Y(\cdot, W)) \rangle_t = \int_0^t A_{\phi^2}(Y(s\Lambda, W))ds. \quad (5)$$

Beside these martingale tools, we will also make use of the following formula for the variance $\text{Var}(\Sigma_\phi(Y(t\Lambda, W)))$ of $\Sigma_\phi(Y(t\Lambda, W))$, $W \subset \mathbb{R}^d$ compact, convex and with interior points, established in full generality in [21] in order to calculate the variance of the limit random variable of our non-Gaussian limit theorem.

Proposition 2 *For any non-degenerate translation-invariant Λ on \mathcal{H} and ϕ as in (3), we have*

$$\text{Var}(\Sigma_\phi(Y(t\Lambda, W))) = \int_{[W]} \zeta^2(\bar{\mathbf{n}}(H)) \int_{W \cap H} \int_{W \cap H} \frac{1 - \exp(-t\Lambda([xy]))}{\Lambda([xy])} dx dy \Lambda(dH). \quad (6)$$

Let us further recall from [21] that the variance of the total edge length of a stationary and isotropic STIT tessellation $Y(t\Lambda_{\text{iso}}, W_R)$ in the plane behaves asymptotically like

$$\text{Var}(\text{Vol}_1(Y(t\Lambda_{\text{iso}}, W_R))) \sim \pi \text{Vol}_2(W) R^2 \log R, \quad R \rightarrow \infty, \quad (7)$$

where W is again a compact convex set and RW stands for W dilated by a factor $R > 0$. Indeed, this can be seen from the general statement from Proposition 2 combined with integral-geometric tools. Note that the asymptotic variance expression for the total edge length is independent of t . However, for all space dimensions > 2 , the surface density t enters the asymptotic variance expression as shown in detail in [21].

We can now turn to the statement of our functional limit theorems, from which Theorems 1 and 2 are direct consequences of.

Theorem 4 *For each $s_0 > 0$ the centred surface increment process*

$$\left(\mathcal{S}_{s_0, t}^{R, W} := \frac{1}{R^{d/2}} [\bar{\Sigma}_\phi(Y(t, W_R)) - \bar{\Sigma}_\phi(Y(s_0, W_R))] \right)_{t \in [s_0, 1]},$$

converges in law, as $R \rightarrow \infty$, on the space $\mathcal{D}[s_0, 1]$ of right continuous functions with left-hand limits (càdlàg) on $[s_0, 1]$, endowed with the usual Skorokhod topology [3, Chap. 3, Sec. 14], to a time-changed Wiener process

$$t \mapsto \mathcal{W}_{V_W(\phi, \Lambda) \int_{s_0}^t s^{1-d} ds},$$

where $\mathcal{W}_{(\cdot)}$ is the standard Wiener process and $V_W(\phi, \Lambda)$ is given by (9) or alternatively (17) below. In particular,

$$\mathcal{S}_{s_0, 1}^{R, W} = \frac{1}{R^{d/2}} [\bar{\Sigma}_\phi(Y(1, W_R)) - \bar{\Sigma}_\phi(Y(s_0, W_R))]$$

converges in law to $\mathcal{N}(0, V_W(\phi, \Lambda) \int_{s_0}^1 s^{1-d} ds)$, a normal distribution with mean 0 and variance $V_W(\phi, \Lambda) \int_{s_0}^1 s^{1-d} ds$.

We consider now the functional convergence of the total length process in the planar case. Write

$$\tau(s, R) := \exp([\log R - \log \log R](s - 1)) = R^{s-1} (\log R)^{1-s}$$

and define the *total* length process

$$\mathcal{L}_s^{R, W} := \frac{1}{R \sqrt{\log R}} \bar{\Sigma}_\phi(Y(\tau(s, R), W_R)), \quad s \in [0, 1].$$

Theorem 5 *The total length process $(\mathcal{L}_s^{R, W})_{s \in [0, 1]}$ converges in law, as $R \rightarrow \infty$, on the space $\mathcal{D}[0, 1]$ of càdlàg functions on $[0, 1]$ endowed with the usual Skorokhod topology, to $(\sqrt{V_W(\phi, \Lambda)} \mathcal{W}_s)_{s \in [0, 1]}$ where, again, $\mathcal{W}_{(\cdot)}$ stands for the standard Wiener process.*

3 Proofs

After having rephrased some background material on STIT tessellations in the previous section, we are now prepared to present the proofs of our limit theorems. Let us briefly recall that we will deal with a fixed translation-invariant hyperplane measure Λ and that for this reason we will write $Y(t)$ instead of $Y(t\Lambda)$ without confusion. Moreover, we fix some compact and convex set $W \subset \mathbb{R}^d$ having interior points and write $W_R = RW$ for W dilated with a factor $R > 0$. Moreover, recall that the face functionals we are dealing with have the representation (3), that $\Sigma_\phi(Y(t))$ was defined in (4) and that the bar notation $\bar{\Sigma}_\phi(Y(t))$ stands for the centred version $\Sigma_\phi(Y(t)) - \mathbb{E}\Sigma_\phi(Y(t))$. We start now with the

Proof of Theorem 4. Notice first that, because of $\Lambda([W_R]) = R\Lambda([W])$,

$$\begin{aligned} \frac{1}{R^d} A_{\phi^2}(Y(1, W_R)) &= \frac{1}{R} \int_{[W_R]} \frac{1}{R^{d-1}} \zeta^2(\vec{n}(H)) \sum_{f \in \text{Cells}(Y(1, W_R) \cap H)} \text{Vol}_{d-1}^2(f) \Lambda(dH) \\ &= \int_{[W]} \frac{1}{R^{d-1}} \zeta^2(\vec{n}(H)) \sum_{f \in \text{Cells}(Y(1, W_R) \cap RH)} \text{Vol}_{d-1}^2(f) \Lambda(dH). \end{aligned} \quad (8)$$

We claim that, upon letting $R \rightarrow \infty$, this converges in probability to

$$\begin{aligned} V_W(\phi, \Lambda) &:= \int_{[W]} \zeta^2(\vec{n}(H)) \text{Vol}_{d-1}(W \cap H) \frac{\mathbb{E} \text{Vol}_{d-1}^2(\text{TypicalCell}(Y(1) \cap H))}{\mathbb{E} \text{Vol}_{d-1}(\text{TypicalCell}(Y(1) \cap H))} \Lambda(dH) \\ &= \text{Vol}_d(W) \int_{\mathcal{S}_{d-1}} \zeta^2(u) \frac{\mathbb{E} \text{Vol}_{d-1}^2(\text{TypicalCell}(Y(1) \cap u^\perp))}{\mathbb{E} \text{Vol}_{d-1}(\text{TypicalCell}(Y(1) \cap u^\perp))} \mathcal{R}(du), \end{aligned} \quad (9)$$

where u^\perp is the orthogonal complement of $u \in \mathcal{S}_{d-1}$ and \mathcal{R} is the directional distribution of the stationary STIT tessellations $Y(t)$ as given in (1). To see it, recall that $Y(1) \cap RH$ is a STIT tessellation in RH for each $R > 0$ and $H \in \mathcal{H}$. Thus, applying [17, (4.6) and Thm. 4.1.3] to this tessellation and the fact that $\mathbb{E} \text{Vol}_d(\text{TypicalCell}(Y(1) \cap u^\perp))$ is the same as the inverse cell intensity of the tessellation $Y(1) \cap u^\perp$, see (10.4) ibidem, we have

$$\begin{aligned} &\lim_{R \rightarrow \infty} \frac{1}{R^{d-1}} \mathbb{E} \sum_{f \in \text{Cells}(Y(1, W_R) \cap RH)} \text{Vol}_{d-1}^2(f) \\ &= \text{Vol}_{d-1}(W \cap H) \frac{\mathbb{E} \text{Vol}_{d-1}^2(\text{TypicalCell}(Y(1) \cap H))}{\mathbb{E} \text{Vol}_{d-1}(\text{TypicalCell}(Y(1) \cap H))}. \end{aligned} \quad (10)$$

Next, we observe that $Y(1, W_R) \cap RH \stackrel{D}{=} Y(1) \cap R \cdot_H (H \cap W)$, where \cdot_H is the scalar multiplication relative in H , that is to say $H \ni R \cdot_H x = p_H(0) + R(x - p_H(0))$, $x \in H$ with p_H standing for the orthogonal projection on H . Thus, applying the recently

developed strong mixing and tail triviality theory for STIT tessellations, especially [12, Thm. 2], noting that tail trivial stationary processes are ergodic [6, Prop. 14.9] and then using the standard multidimensional ergodic theorem, see e.g. Corollary 14.A5 ibidem, to $\frac{1}{R^{d-1}} \sum_{f \in \text{Cells}(Y(1) \cap R \cdot_H (H \cap W))} \text{Vol}_{d-1}^2(f)$, we get from (10) that

$$\begin{aligned} & \lim_{R \rightarrow \infty} \frac{1}{R^{d-1}} \sum_{f \in \text{Cells}(Y(1, W_R) \cap RH)} \text{Vol}_{d-1}^2(f) \\ &= \text{Vol}_{d-1}(W \cap H) \frac{\mathbb{E} \text{Vol}_{d-1}^2(\text{TypicalCell}(Y(1) \cap H))}{\mathbb{E} \text{Vol}_{d-1}(\text{TypicalCell}(Y(1) \cap H))} \end{aligned}$$

in probability. Putting this together with (8) and integrating over $[W]$ yields

$$\lim_{R \rightarrow \infty} \frac{1}{R^d} A_{\phi^2}(Y(1, W_R)) = V_W(\phi, \Lambda) \quad \text{in probability,} \quad (11)$$

as required. Note now that by the scaling properties of $Y(s, W_R)$ and ϕ^2 for $s > 0$ we have

$$\frac{1}{R^d} A_{\phi^2}(Y(s, W_R)) \stackrel{D}{=} \frac{1}{R^d s^{2d-1}} A_{\phi^2}(Y(1, W_{sR})) \stackrel{D}{=} \frac{1}{s^{d-1}} \frac{1}{(Rs)^d} A_{\phi^2}(Y(1, W_{sR})). \quad (12)$$

Thus, combining (11) with the scaling relation (12) we get

$$\lim_{R \rightarrow \infty} \frac{1}{R^d} A_{\phi^2}(Y(s, W_R)) = \frac{1}{s^{d-1}} V_W(\phi, \Lambda) \quad (13)$$

in probability uniformly in $s \in [s_0, 1]$. This crucial statement puts us now in context of the general martingale limit theory. Indeed, using Proposition 1 we see that $\mathcal{S}_{s_0, s}^{R, W} = \frac{1}{R^{d/2}} [\bar{\Sigma}_\phi(Y(1, W_R)) - \bar{\Sigma}_\phi(Y(s_0, W_R))]$ is a martingale with absolutely continuous predictable quadratic variation process

$$\langle \mathcal{S}_{s_0, \cdot}^{R, W} \rangle_t = \int_{s_0}^t \frac{1}{R^d} A_{\phi^2}(Y(s, W_R)) ds, \quad (14)$$

see [11, Thm. 4.2]. In these terms, (13) yields for each t

$$\lim_{R \rightarrow \infty} \langle \mathcal{S}_{s_0, \cdot}^{R, W} \rangle_t = \int_{s_0}^t \frac{1}{s^{d-1}} V_W(\phi, \Lambda) \quad \text{in probability.} \quad (15)$$

We now want to apply the martingale functional central limit theorem. Whereas this is well known for continuous martingales, we need a version for martingales in the Skorokhod space $\mathcal{D}[s_0, 1]$. In this paper, we will make use of the version formulated as Theorem 2.1 in the survey article [24]. In order to apply this theorem, several conditions have to be checked. Condition (ii.6) in [24, Thm. 2.1] is just (15), whereas condition (ii.4) there is trivially verified, because the predictable quadratic variation $\langle \mathcal{S}_{s_0, \cdot}^{R, W} \rangle$ has no jumps by (14). It remains to check the condition (ii.5) ibidem, which is that the second moment of

the maximum jump $\mathcal{J}(\mathcal{S}_{s_0, \cdot}^{R,W}; 1)$ of the process $(\mathcal{S}_{s_0, s}^{R,W})_{s \in [s_0, 1]}$ goes to 0 as $R \rightarrow \infty$. More precisely,

$$\mathcal{J}(\mathcal{S}_{s_0, \cdot}^{R,W}; 1) = \sup_{s_0 \leq t \leq 1} |\mathcal{S}_{s_0, t}^{R,W} - \mathcal{S}_{s_0, t-}^{R,W}|, \quad \mathcal{S}_{s_0, t-}^{R,W} = \lim_{s \uparrow t} \mathcal{S}_{s_0, s}^{R,W},$$

and we have to check that $\mathbb{E}\mathcal{J}^2(\mathcal{S}_{s_0, \cdot}^{R,W}; 1) \rightarrow 0$, as $R \rightarrow \infty$. To this end, note first that, with probability one, $\mathcal{J}(\mathcal{S}_{s_0, \cdot}^{R,W}; 1)$ is bounded from above by a constant multiple of $R^{-d/2}$ times the $(d-1)$ -th power of the diameter of the largest cell of $Y(s_0, W_R)$. Since the typical cell of $Y(s_0)$ is the same as that of a Poisson hyperplane tessellation with intensity measure $s_0\Lambda$ (see Theorem 1 in [20] or Section 2 above), we conclude by standard properties of Poisson hyperplane tessellations that the expected number of cells in $Y(s_0, W_R)$ with diameter exceeding D is of the order $O(R^d \exp(-D))$, since Λ has been assumed to have its support spanning the whole of \mathbb{R}^d . Summarizing, setting $u = D^{d-1}R^{-d/2}$, we are led to

$$\mathbb{P}(\mathcal{J}(\mathcal{S}_{s_0, \cdot}^{R,W}; 1) > u) = O(R^d \exp(-R^{d/(2d-2)} u^{1/(d-1)})). \quad (16)$$

Clearly, (16) is much more than enough to guarantee that

$$\lim_{R \rightarrow \infty} \mathbb{E}\mathcal{J}^2(\mathcal{S}_{s_0, \cdot}^{R,W}; 1) = 0,$$

which gives the required condition (ii.5) of Theorem 2.1 in [24]. Upon a trivial time change, this theorem yields now the functional convergence in law as stated in our Theorem 4. \square

Before turning to the proof of Theorem 5 we provide an alternative formula for the factor $V_W(\phi, \Lambda)$. Readers not specialized in convex or stochastic geometry could also skip this alternative representation and directly jump to the next paragraph, because Proposition 3 will not be used in the sequel. However, having such a more explicit variance expression could be useful for other purposes and was already used in our subsequent work [23]. We denote, as in [20], by Π the associated zonoid of a Poisson hyperplane tessellation with intensity measure Λ , by Π° its dual body and by \mathcal{R} the directional distribution of the STIT tessellation from (1), see [17] for the precise definition of Π .

Proposition 3 *We have*

$$V_W(\phi, \Lambda) = \text{Vol}_d(W) \frac{(d-1)!}{2^{d-1}} \int_{\mathcal{S}_{d-1}} \zeta^2(u) \text{Vol}_{d-1}((\Pi|u^\perp)^\circ) \mathcal{R}(du), \quad (17)$$

where $\Pi|u^\perp$ stands for the orthogonal projection of Π onto the hyperplane u^\perp , and where the polar body $(\Pi|u^\perp)^\circ$ is considered relative to u^\perp . In the isotropic case, i.e. when $\mathcal{R} = \nu_{d-1}$, the uniform distribution on the unit sphere \mathcal{S}_{d-1} , this reduces to

$$V_W(\phi, \Lambda_{\text{iso}}) = \text{Vol}_d(W) 2^{d-1} \pi^{d-\frac{3}{2}} \Gamma\left(\frac{d+1}{2}\right)^{d-1} \Gamma\left(\frac{d}{2}\right)^{2-d} \int_{\mathcal{S}_{d-1}} \zeta^2(u) \nu_{d-1}(du).$$

In particular for $\zeta \equiv 1$, $W = B_1^d$ the unit ball and $d = 2$ and $d = 3$ we conclude the exact values

$$V_{B_1^2}(\text{Vol}_1, \Lambda_{\text{iso}}) = \pi^2 \quad \text{and} \quad V_{B_1^3}(\text{Vol}_2, \Lambda_{\text{iso}}) = \frac{32}{3} \pi^2.$$

Proof of Proposition 3. At first, [5, Cor. 3.7] provides a general formula for the second moment of the volume of the typical Poisson cell of a stationary Poisson hyperplane tessellation in \mathbb{R}^d . In terms of the zonoid Π it reads

$$\mathbb{E} \text{Vol}_d^2(\text{TypicalCell}(\text{PHT}(\Lambda))) = \frac{d!}{2^d} \frac{\text{Vol}_d(\Pi^o)}{\text{Vol}_d(\Pi)},$$

where we have used formula [17, (4.63)]. Moreover, the mean volume of $\text{TypicalCell}(\text{PHT}(\Lambda))$ is given by

$$\mathbb{E} \text{Vol}_d(\text{TypicalCell}(\text{PHT}(\Lambda))) = \frac{1}{\text{Vol}_d(\Pi)}$$

according to [17, Thm. 10.3.3 and (10.4)]. Using now Eq. (4.61) ibidem and the fact that STIT tessellations have Poisson typical cells and replacing d by $d - 1$ in the last two formulas we obtain (17). The precise value in the stationary and isotropic case can be calculated from the fact that in this case, Π is a ball with a known radius, see [17]. \square

We are now going to present the

Proof of Theorem 5. Note first that

$$\tau(0, R) = \frac{\log R}{R}, \quad \tau(1, R) = 1, \quad \frac{\partial}{\partial s} \tau(s, R) = \tau(s, R)[\log R - \log \log R]. \quad (18)$$

Thus, defining the auxiliary process

$$M_s^{R,W} = M_s := \frac{1}{R\sqrt{\log R - \log \log R}} [\bar{\Sigma}_\phi(Y(\tau(s, R), W_R)) - \bar{\Sigma}_\phi(Y(\tau(0, R), W_R))]$$

and using (4) with $W_R := RW$ and under variable substitution $s := \tau(u, R)$ and $t := s$ with LHS variables corresponding to the notation of (4) and RHS to that used here, we see that, by (18),

$$(M_s)_{s=0}^1 \text{ and } \left(M_s^2 - \int_0^s \frac{\tau(u, R)}{R^2} A_{\phi^2}(Y(\tau(u, R), W_R)) du \right)_{s \in [0,1]}$$

are $\mathfrak{S}_{\tau(s,R)}$ -martingales. In particular, see [11, Thm. 4.2], the predictable quadratic variation process $\langle M \rangle_s$ is given by

$$\langle M \rangle_s = \int_0^s \frac{\tau(u, R)}{R^2} A_{\phi^2}(Y(\tau(u, R), W_R)) du, \quad s \in [0, 1]. \quad (19)$$

Repeating the argument leading to (13) we see that

$$\lim_{R \rightarrow \infty} \frac{\tau(s, R)}{R^2} A_{\phi^2}(Y(\tau(s, R), W_R)) = V_W(\phi, \Lambda) \quad (20)$$

in probability, uniformly in $s \in [0, 1]$. Note that the uniformity in s comes, as in the case of (13), from the relation (12) implying that, in distribution, all instances of the LHS for different values of s are just scaling instances of the same object $\tilde{R}^{-2}A_{\phi^2}(Y(1, W_{\tilde{R}}))$ for $\tilde{R} = R/\tau(s, R)$ and thus, in terms of the considered convergence in probability to a deterministic limit, we are just dealing with a single asymptotic statement. Consequently, by (20) and in full analogy to (15),

$$\lim_{R \rightarrow \infty} \langle M \rangle_s = \int_0^s V_W(\phi, \Lambda) du = s V_W(\phi, \Lambda) \quad \text{in probability.} \quad (21)$$

Thus, we are again in a position to apply [24, Thm 2.1] yielding the functional convergence in law, as $R \rightarrow \infty$, in $\mathcal{D}[0, 1]$ of $(M_s)_{s \in [0, 1]}$ to $(\sqrt{V_W(\phi, \Lambda)} \mathcal{W}_s)_{s \in [0, 1]}$. Indeed, condition (ii.6) there is just (21), condition (ii.4) is trivial in view of (19), whereas the condition (ii.5) is verified by noting that, with probability one, $\mathcal{J}(M; 1) = \frac{1}{R\sqrt{\log R}} O(R \text{diam}(W)) = O(1/\sqrt{\log R})$, so that in particular $\lim_{R \rightarrow \infty} \mathbb{E} \mathcal{J}^2(M; 1) = 0$, as required. Denoting now by $C^{R, W}$ the *correction term* $(R\sqrt{\log R})^{-1} \bar{\Sigma}_\phi(Y(\tau(0, R), W_R))$ such that

$$\mathcal{L}_s^{R, W} = C^{R, W} + \sqrt{\frac{\log R - \log \log R}{\log R}} M_s,$$

noting that $\log R - \log \log R \sim \log R$ and that, by the scaling property of STIT tessellations and by (7),

$$\text{Var}(C^{W, R}) = O([R^{-2}(\log R)^{-1}][R^2/(\log R)^2][(\log R)^2(\log \log R)]) = O(\log \log R / \log R),$$

we see that the processes M_s and $\mathcal{L}_s^{R, W}$ are asymptotically equivalent in $\mathcal{D}[0, 1]$ as $R \rightarrow \infty$. This completes the proof of Theorem 5. \square

In the context of proof of Theorem 5 it should be remarked that the ‘negligible correction term’ $C^{R, W}$ has its variance of order $O(\log \log R / \log R)$ and thus indeed tending to 0, but extremely slowly. Consequently, although the Gaussian CLT holds for $\mathcal{L}_0^{R, W}$, it is quite natural to expect that the convergence rates are extremely slow, conjecturedly logarithmic. This is due to the fact that dimension 2 is the largest dimension (critical dimension) where the Gaussian limits are still present. In dimensions 3 and higher there is no Gaussian CLT and the ‘correction term’ analogous to $C^{R, W}$ will turn out order-determining rather than negligible, as shown by Theorem 3.

Proof of Theorem 3. Even if in the formulation of the theorem we have used the volume functional, we will show the statement in a more general context, where the volume $\text{Vol}_{d-1}(1, W_R)$ is replaced by a general face functional $\Sigma_\phi(1, W_R)$ satisfying (3).

We claim that the argument from the proof of Theorem 5 cannot be repeated for $d > 2$. Intuitively, this is due to the fact that for $d > 2$ the variance order of $\bar{\Sigma}_\phi(Y(1, W_R))$ is $O(R^{2(d-1)})$, see below, whereas the variance order of the increment $\bar{\Sigma}_\phi(Y(1, W_R)) - \bar{\Sigma}_\phi(Y(s_0, W_R))$, with some time instant $0 < s_0 < t$, is $O(R^d)$ as seen by Theorem 4.

Hence, for $d > 2$ we conclude that even the very first faces born in the MNW-cell-division process already bring a non-negligible contribution to the overall variance. Thus, we cannot split the whole STIT construction into the *warm-up phase* ($t \in [0, R^{-1} \log R]$ for $d = 2$) with negligible variance contribution and the *proper phase* unfolding already in a typical STIT environment. In fact, the claim is that the CLT does not hold for STIT surface functionals in dimension greater than 2!

Recall that we do not show this fact in full generality for all non-degenerate hyperplane measures Λ and all windows W , but restrict ourself to a particular case, where Λ is given by (2) and where $W = [0, 1]^d$. To see the non-Gaussianity, observe first that, by the scaling property of STIT tessellations,

$$R^{-(d-1)} \bar{\Sigma}_\phi(Y(1, W_R)) \stackrel{D}{=} \bar{\Sigma}_\phi(Y(R, W)), \quad (22)$$

which implies that the variance $\text{Var}(\Sigma_\phi(Y(1, W_R)))$ is of order $O(R^{2(d-1)})$. Indeed, this follows directly from the special form (3) of the face functional ϕ and the scaling relation $Y(1, W_R) \stackrel{D}{=} RY(R, W)$. Further, recall that by (4) the process $R \mapsto \bar{\Sigma}_\phi(Y(R, W))$ is a square-integrable martingale with absolutely continuous predictable quadratic variation process given in (5) and, moreover, by Proposition 2 we have

$$\text{Var}(\Sigma_\phi(Y(R, W))) = \int_{[W]} \zeta^2(\vec{\mathbf{n}}(H)) \int_{W \cap H} \int_{W \cap H} \frac{1 - \exp(-R\Lambda([xy]))}{\Lambda([xy])} dx dy \Lambda(dH),$$

which is bounded uniformly in R . Consequently, by the martingale convergence theorem, there exists a centered square-integrable random variable $\Xi(W)$ such that

$$\Xi(W) = \lim_{R \rightarrow \infty} \bar{\Sigma}_\phi(Y(R, W))$$

a.s. and in L^2 and, moreover,

$$\begin{aligned} \text{Var}(\Xi(W)) &= \lim_{R \rightarrow \infty} \text{Var}(\Sigma_\phi(Y(R, W))) \\ &= \int_{[W]} \zeta^2(\vec{\mathbf{n}}(H)) \int_{W \cap H} \int_{W \cap H} \frac{1}{\Lambda([xy])} dx dy \Lambda(dH). \end{aligned} \quad (23)$$

Using now (22) we readily conclude that

$$R^{-(d-1)} \bar{\Sigma}_\phi(Y(1, W)) \implies \Xi(W)$$

as $R \rightarrow \infty$, where \implies stands for convergence in distribution.

We show now that the variable $\Xi(W)$ cannot be Gaussian. To see it, consider the event \mathcal{E}_N , $N > 0$, that only hyperplanes orthogonal to e_1 have been born during the time $[0, 1]$ in the MNW-construction and that their number exceeds N . Observe that, in view of the form (2) of Λ , $\mathbb{P}(\mathcal{E}_N) = \exp(-d) \sum_{k=N+1}^{\infty} \frac{1}{k!}$ and thus

$$\log \mathbb{P}(\mathcal{E}_N) = -\Theta(N \log N), \quad (24)$$

where by $\Theta(\cdot)$ we mean something bounded both from below and above by multiplicities of the argument, i.e. $\Theta(\cdot) = O(\cdot) \cap \Omega(\cdot)$ in the usual Landau notation. Further, given the fixed collection of all hyperplanes H_1, \dots, H_k , $k > N$, born at times between 0 and 1, on the event \mathcal{E}_N , we see that the conditional law of $\Xi(W)$ coincides with that of $k - d$ plus the sum of independent copies ξ_1, \dots, ξ_{k+1} of $\Xi(W_1), \dots, \Xi(W_{k+1})$ respectively, where W_j , $j = 1, \dots, k+1$, are the parallelepipeds into which W is partitioned by H_1, \dots, H_k . Note that the extra k above is the sum of $(d-1)$ -volumes of $W \cap H_i$, whereas $-d = -\mathbb{E}\Sigma_\phi(Y(1, W))$ is the centring term. Since $\text{Var}(\xi_1 + \dots + \xi_{k+1}) = \sum_{j=1}^{k+1} \text{Var} \Xi(W_j)$, which is bounded from above by $\text{Var}(\Xi(W))$ in view of (23), by Chebyshev's inequality we get

$$\mathbb{P}(\xi_1 + \dots + \xi_{k+1} \geq -2\sqrt{\text{Var}(\Xi(W))}) \geq 1 - \frac{\text{Var}(\xi_1 + \dots + \xi_{k+1})}{4 \text{Var}(\Xi(W))} \geq 3/4.$$

Thus, in view of (24)

$$\mathbb{P}(\Xi(W) > N) \geq \frac{3}{4} \mathbb{P}\left(\mathcal{E}_{N+2\sqrt{\text{Var}(\Xi(W))+d}}\right) = \exp(-\Theta(N \log N)).$$

Since Gaussian variables exhibit tail decay of the order $\exp(-\Theta(N^2))$, the random variable $\Xi(W)$ cannot be Gaussian, which completes our argument. \square

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